CS109 Quiz 1 Review Session

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Today's Plan

Logistics

Strategies

Conceptual Review

Practice Problems

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Logistics

- Wednesday, 11:00AM Pacific until Friday, April 23, 10:00AM Pacific
- Don't pull an all-nighter! You have 48 hours, but we think you'll spend 1-2 hours plus time to typeset answers
- Open-book, closed-people. See website.
- We can't answer quiz or problem set questions during the testing period.

Coverage

Lectures 1-6 inclusive, Problem Sets 1 and 2.

A non-exhaustive topic list:

- Counting
  - Sum Rule, Product Rule
  - Inclusion-Exclusion
  - Pigeonhole Principle
  - Permutations, Combinations, and Buckets
- Probability
  - Event space, sample space
  - Axioms of Probability
  - Conditional Probability, Chain Rule, Law of Total Probability
  - Bayes' Theorem
  - Independence
- Random Variables
  - PMF, CDF
  - Expectation

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Strategies

- Write down what you *want to find* and what you *already know*.
- *Justify* your answers in math and English.
  - We aren't looking for an essay, but we need enough explanation so we can follow your thought process.
  - This helps us give you partial credit! Help us help you.
  - Justification is more important than the final answer.
- *Define* any variable that you use. (If you don't know what a variable stands for, neither do we.)
- Typeset your answers or write neatly. (Don't make us guess your handwriting!)

Strategies

- *Read* the problem carefully. Single words like "distinct" or "independent" make a difference!

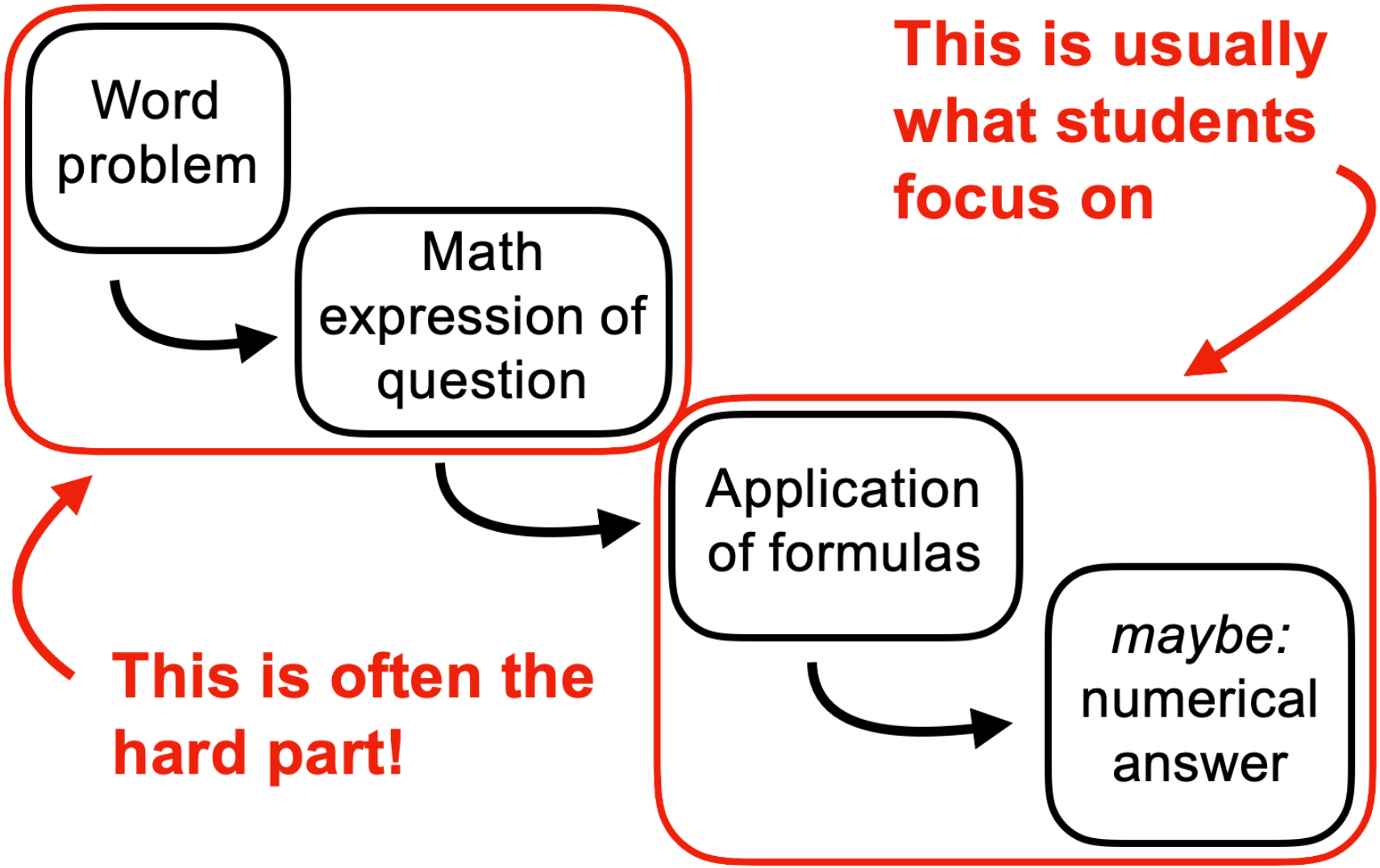
Strategies

- *Read* the problem carefully. Single words like "distinct" or "independent" make a difference!
  - Counting: What is distinct? Which orders do I care about? Can I come up with a generative process?
  - Probability: Are events independent? Definition of conditional probability? Bayes? Law of total probability? What's a "success"? What's the event space? What's the sample space?
  - Random variables: What values does it take on?

Strategies

- *Read* the problem carefully. Single words like "distinct" or "independent" make a difference!
  - Counting: What is distinct? Which orders do I care about? Can I come up with a generative process?
  - Probability: Are events independent? Definition of conditional probability? Bayes? Law of total probability? What's a "success"? What's the event space? What's the sample space?
  - Random variables: What values does it take on?
- "Open book" doesn't mean "don't study."
  - When in doubt, refer to the course materials.
  - Use course materials as a tool to refresh your memory, not as a tool to learn new skills during the testing period.
- Don't memorize formulas. *Understand them.*

Solving a CS109 Problem



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Counting

- **Sum Rule of Counting:** If outcome of an experiment comes from either Set A or Set B, where $A \cap B = \varnothing$, then the experiment has $|A| + |B|$ outcomes.
  - Example: I can choose to wear one of 3 t-shirts or one of 5 collared shirts. How many total options?
- **Inclusion-Exclusion Principle:** Same as above, but no constraints about $A \cap B$. Number of outcomes is $|A| + |B| - |A \cap B|$.
  - Example: I can choose to wear one of 3 fancy shirts or one of 5 collared shirts. 2 of the fancy shirts are also collared. How many total options?

Counting

- **Product Rule of Counting:** If an experiment has two parts, where the first part's outcomes are from Set A, and the second part's outcomes are from Set B, then the experiment has $|A||B|$ outcomes.
  - Example: I can choose to wear one of 3 t-shirts and one of 5 pants. How many outfits?
- **General Principle of Counting:** If an experiment has r steps, where step i has n_i outcomes for all $i = 1, 2, \ldots, r$, then the experiment has $(n_1)(n_2)\ldots(n_r) = \prod_{i=1}^{r} n_i$ outcomes.
- **Pigeonhole Principle:** If m objects are placed into n buckets, then at least one bucket has $\lceil m/n \rceil$ objects.

Permutations

- **All distinct:** If n objects are all distinct, there are $n!$ permutations.
  - Example: how many different orderings of the alphabet
- **Semi distinct:** If we have n objects where n_1 are indistinct, n_2 are indistinct, ..., n_r are indistinct, then there are $\frac{n!}{n_1!n_2!...n_r!}$ unique permutations.
  - Conceptually: pretend all n objects are distinct, then divide by overcounted permutations.
  - Example: how many orderings of 3 Coke Zeros and 4 Diet Cokes

Combinations

- How many unordered ways to select k objects from n distinct objects:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  - Example: How many ways to pick 4 crayons from a box of 10
- How many ways to put n distinct objects into r groups such that each group has size n_i and all n objects are put in a group, i.e. $\sum_{i=1}^{r} n_i = n$:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

  - Example: How many ways to pick 4 crayons for me and 6 for my friend from a box of 10 distinct crayons?
    - Answer: $\binom{10}{4,6}$
  - Example: How many ways to pick 3 crayons for me and 2 for my friend from a box of 10 distinct crayons?
    - Answer: $\binom{10}{3,2,5}$ - must implicitly select 5 crayons to be given to neither person

Buckets

- How many ways to put n **distinct** objects into r buckets: r^n
  - Example: hashing n strings into r buckets
- How many ways to put n **indistinct** objects into r buckets: $\binom{n+r-1}{r-1}$
  - Group n indistinct objects into $r-1$ indistinct dividers
  - Example: assign indistinct children to different ice cream flavors

Axioms of Probability

1. $0 \leq P(E) \leq 1$: *a probability must be between 0 and 1*
2. $P(S) = 1$: *the sample space must be "guaranteed" to happen*
3. If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollaries

- $P(E^C) + P(E) = 1$
- If $E \subseteq F$, then $P(E) \leq P(F)$.
- $P(E \cup F) = P(E) + P(F) - P(EF)$

Conditional Probability

- **Conditional Probability Definition:**

$$P(E|F) = \frac{P(EF)}{P(F)}$$

- Rearrange that to get **Chain Rule:**

$$P(EF) = P(F)P(E|F)$$

- **Law of Total Probability:**

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

- Remember: when we say $P(E|F)$, we are in the world where we *know* that event F happens.
- Example: how likely am I to ace the quiz if I attend the review session?

Bayes' Theorem

- **Bayes' Theorem Definition**

$$\overbrace{P(F|E)}^{\text{posterior}} = \frac{\overbrace{P(E|F)}^{\text{likelihood}}\overbrace{P(F)}^{\text{prior}}}{\underbrace{P(E)}_{\text{normalization constant}}}$$

- Lets us go from $P(E|F)$ to $P(F|E)$
- Often expanded with Chain Rule:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

- Partial example: let's say we know the probability that someone aced the quiz given that they attended this review session. We want to find the probability that someone attended the review session given that they aced the quiz.

Conditioning + Axioms

For events A, B, and E, we can condition on E, and the key axioms still hold:

$$\text{Axiom 1} : 0 \leq P(A|E) \leq 1$$

$$\text{Corollary 1} : P(A|E) + P(A^C|E) = 1$$

$$\text{Transitivity} : P(AB|E) = P(BA|E)$$

$$\text{Chain Rule} : P(AB|E) = P(B|E)P(A|BE)$$

$$\text{Bayes' Theorem} : P(A|BE) = \frac{P(B|AE)P(A|E)}{P(B|E)}$$

Independence and Conditional Independence

Independence:

- Events A and B are independent if $P(AB) = P(A)P(B)$.
- Events A and B are independent if $P(A|B) = P(A)$
- More than 2 events are independent: see lecture notes/slides
- Not the same as mutual exclusion!

Conditional Independence:

- A and B are conditionally independent given E if $P(AB|E) = P(A|E)P(B|E)$
- A and B are conditionally independent given E if $P(A|BE) = P(A|E)$

Independence can change with conditioning. If A and B are independent, that doesn't necessarily mean that A and B are independent given E.

Independence vs. Conditional Independence: Example

- Normally, the event that I am carrying a large bag of candy on me (event E) and that I am in front of a large group of people (event F) are independent events. $P(E) = 0.001$ and $P(F) = 0.2$.
- What's the probability I have candy and I'm in front of people?
  - $P(EF) = P(E)P(F) = 0.0002$
- Given that it is review day (event G), we have that $P(E|G) = 0.2$, $P(F|G) = 0.9$, and $P(EF|G) = 0.8$. Are E and F conditionally independent given G?
  - $P(E|G)P(F|G) = (0.2)(0.9) = 0.18 \neq P(EF|G)$
  - Thus, E and F are not conditionally independent given G.
- **Again, independence can change with conditioning.**

Independence and Mutual Exclusion

- Two events can be independent, but not mutually exclusive.
  - E.g. What is the probability that it's raining and I am eating a pomegranate? Both events are independent, but they can both occur simultaneously. Sometimes it's raining, sometimes I'm eating a pomegranate, and sometimes it's raining and I'm eating a pomegranate.
- Two sets of events can be mutually exclusive but not necessarily independent.
  - I am either studying or sleeping; these are mutually exclusive events. However, how much I study affects how much I sleep, so therefore they are not necessarily independent.
- **Call back to definitions** whenever trying to prove independence.

Random Variables

- X is a random variable. $X = 2$ is an event. $P(X = 2)$ is a probability.
  - Make sure to understand the difference!
- **Probability Mass Function** of a discrete random variable: $P(X = x) = p(x) = p_X(x)$
  - Must have $\sum_x P(X = x) = 1$
- **Cumulative Distribution Function**: $F(a) = F_X(a) = P(X \leq a)$.
  - For a discrete RV X, the CDF is $F(a) = P(X \leq a) = \sum_{\{x|x \leq a\}} p(x)$
- Examples of random variables:
  - Value of a dice roll
  - Number of aces drawn from a deck

Expectation

Definition:

$$E[X] = \sum_{x:p(x)>0} x \cdot P(X = x)$$

Linearity of Expectation:

$$E[aX + b] = aE[X] + b$$

Expectation of Sum:

$$E[X + Y] = E[X] + E[Y]$$

Law of the Unconscious Statistician:

$$E[g(X)] = \sum_{x} g(x)p(x)$$

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GRE Testing (Extra Problems #1, Q1)

- A computerized test tries to estimate whether or not a student knows geometry based on whether they answer the first geometry question correctly.
- The test's prior belief that the student knows geometry is q.
- Given that a student knows geometry, we know the probability that they solve the first question correctly is p_1.
- Given that a student does not know geometry, the probability that the solve the first questions correctly is p_2.
- The student solves the first question correctly. What is the probability that they know geometry.

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- Given that a student does not know geometry, the probability that the solve the first questions correctly is p_2.
- The student solves the first question correctly. What is the probability that they know geometry.
- **First, define events and write everything we know in terms of those events.**

GRE Testing (Extra Problems #1, Q1)

- A computerized test tries to estimate whether or not a student knows geometry based on whether they answer the first geometry question correctly.
  - C: event that student knows geometry
  - S: event that student solves first question correctly
- The test's prior belief that the student knows geometry is q.
  - $P(C) = q$, $P(C^c) = 1 - q$
- Given that a student knows geometry, we know the probability that they solve the first question correctly is p_1.
  - $P(S|C) = p_1$
- Given that a student does not know geometry, the probability that the solve the first questions correctly is p_2.
  - $P(S|C^c) = p_2$
- The student solves the first question correctly. What is the probability that they know geometry.
  - Want to find $P(C|S)$

GRE Testing (Extra Problems #1, Q1)

Facts:

- $P(C) = q$
- $P(C^c) = 1 - q$
- $P(S|C) = p_1$
- $P(S|C^c) = p_2$

Want to find $P(C|S)$. We know $P(S|C)$. Let's use Bayes' Theorem:

$$\begin{aligned} P(C|S) &= \frac{P(S|C)P(C)}{P(S)} \\ &= \frac{P(S|C)P(C)}{P(S|C)P(C) + P(S|C^c)P(C^c)} \\ &= \frac{p_1 q}{p_1 q + p_2(1-q)} \end{aligned}$$

Conditional Probabilities: Corrupt Hot-Dog Judges

- Corrupted by their power, the judges running America's Next Top Hot Dog Eater have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability $\frac{1}{3}$, independent of what happens in earlier episodes. Suppose that $\frac{1}{4}$ of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.
- a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?
- b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

Conditional Probabilities: Corrupt Hot-Dog Judges

- B : Event a particular contestant bribed the judges, $P(B) = \frac{1}{4}$
- S_1 : Event a particular contestant stay during the first episode
- S_2 : Event a particular contestant stay during the second episode
- $P(S_i|B) = 1, P(S_i|B^C) = \frac{1}{3}$
- a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?
  - Get $P(S_1)$ by using the Law of Total Probability

$$\begin{aligned} P(S_1) &= P(S_1|B)P(B) + P(S_1|B^C)P(B^C) \\ &= 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} \\ &= \frac{1}{2} \end{aligned}$$

Conditional Probabilities: Corrupt Hot-Dog Judges

- B : Event a particular contestant bribed the judges, $P(B) = \frac{1}{4}$
- S_1 : Event a particular contestant stay during the first episode
- S_2 : Event a particular contestant stay during the second episode
- $P(S_i|B) = 1, P(S_i|B^C) = \frac{1}{3}$
- b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?
  - $P(S_1 \cap S_2) = P(S_1 \cap S_2|B)P(B) + P(S_1 \cap S_2|B^C)P(B^C)$ (Law of Total Probability)
  - Staying in episodes are conditionally independent given whether she bribed the judges

$$
\begin{aligned}
P(S_1 \cap S_2) &= P(S_1|B)P(S_2|B)P(B) + P(S_1|B^C)P(S_2|B^C)P(B^C) \\
&= 1^2 \cdot \frac{1}{4} + \frac{1}{3}^2 \cdot \frac{3}{4} \\
&= \frac{1}{3}
\end{aligned}
$$

Conditional Probabilities: Corrupt Hot-Dog Judges

- c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

Conditional Probabilities: Corrupt Hot-Dog Judges

- c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)}$
  - $P(S_1 \cap S_2^C) = P(S_1|B)P(S_2^C|B)P(B) + P(S_1|B^C)P(S_2^C|B^C)P(B^C) = 1 \cdot 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{6}$
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)} = \frac{1/6}{1/2}$

Conditional Probabilities: Corrupt Hot-Dog Judges

- c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)}$
  - $P(S_1 \cap S_2^C) = P(S_1|B)P(S_2^C|B)P(B) + P(S_1|B^C)P(S_2^C|B^C)P(B^C) = 1 \cdot 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{6}$
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)} = \frac{1/6}{1/2}$
- d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judge?

Conditional Probabilities: Corrupt Hot-Dog Judges

- c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)}$
  - $P(S_1 \cap S_2^C) = P(S_1|B)P(S_2^C|B)P(B) + P(S_1|B^C)P(S_2^C|B^C)P(B^C) = 1 \cdot 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{6}$
  - $P(S_2^C|S_1) = \frac{P(S_2^C \cap S_1)}{P(S_1)} = \frac{1/6}{1/2}$
- d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judge?
  - $P(B|S_1) = \frac{P(S_1|B)P(B)}{P(S_1)} = \frac{(1)(1/4)}{1/2} = \frac{1}{2}$

Harry Potter

- There are 8 Harry Potter films, but I only want to watch 3. Since I've seen them all so many times, I can watch any 3 in any order. For example, if I chose movies A, B, and C, I could watch them ordered as ABC, CBA, etc. How many ways are there for me to watch 3 Harry Potter Movies?

Harry Potter

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- We consider this a two-part experiment: first we pick 3 movies, then we order them.
  - $\binom{8}{3}$ ways to choose 3 movies.
  - 3! ways to order the 3 movies that I chose.
- Answer: $\binom{8}{3}3!$

Rolling Dice (Extra Problems #2, Q5)

- Robin Hood and Maid Marian are playing a game where they roll fair 6-sided dice until they both roll the same number.
- What's the probability they roll the dice at most 5 times?

Rolling Dice (Extra Problems #2, Q5)

- Robin Hood and Maid Marian are playing a game where they roll fair 6-sided dice until they both roll the same number.
- What's the probability they roll the dice at most 5 times?
- Let E be the event where both people roll the same number.
- $P(E) = \frac{1}{6}$, $P(E^C) = 1 - \frac{1}{6} = \frac{5}{6}$
- Let X be a random variable for the number of dice rolls until the end of the game
- $P(X = i) = \frac{5}{6}^{(i-1)} \frac{1}{6}$ for $1 \leq i \leq 5$
- $P(X \leq 5) = \Sigma_{i=1}^{5} P(X = i) = \Sigma_{i=1}^{5} \frac{5}{6}^{(i-1)} \frac{1}{6}$

You can do it!

Good luck!