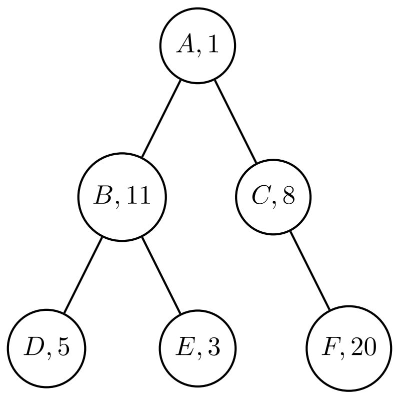
Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Covering a Tree

For any undirected graph $G = (V, E)$, we define a *vertex cover* of the graph to be a set of vertices that includes at least one endpoint of every edge of the graph. Now, suppose you are given a tree graph $T = (V, E)$, where each vertex v is assigned a weight $w(v)$ and the edges are unweighted. Devise a dynamic programming algorithm to compute the minimum weight vertex cover (MWCV) of T, where the weight of a vertex cover is the sum of the weights of its vertices.

As an example, consider the tree depicted below, where each node is formatted as $(v, w(v))$:



In the tree above, the vertex cover that contains the smallest number of vertices is $\{B, C\}$ with weight 19. However, what we want is the minimum weight vertex cover, which is $\{A, C, D, E\}$ with weight 17.

2 LP Canonical Form

Linear Program. A *linear program* is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^n$ be the set of variables and $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. The canonical form of a linear program is

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

where $x \geq 0$ means that every entry of the vector x is greater than or equal to 0.

Any linear program can be written in canonical form. Let's check this is the case:

(i) What if the objective is maximization?

(ii) What if you have a constraint $Ax \leq b$?

(iii) What about $Ax = b$?

(iv) What if the constraint is $x \leq 0$?

(v) What about unconstrained variables $x \in \mathbb{R}$?

(vi) What if the objective is $\min \max\{c_1^\top x, c_2^\top x\}$?

3 Linear Programming Basics

Plot the feasible region and identify the optimal solution for the following linear program.

$$\text{maximize } 5x + 3y$$
$$s.t. \quad 5x - 2y \geq 0, \quad x + y \leq 7, \quad x \leq 5, \quad x \geq 0, \quad y \geq 0$$

Suppose we want to maximize

$$\min\{5x, 3y\}$$

instead, subject to the same constraints. Describe how we can modify the LP to solve this problem by changing the objective, adding one new variable, and adding two new constraints.

4 Understanding convex polytopes

An equivalent formulation of linear programming is as follows

$$(\mathcal{P}) = \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b. \end{cases}$$

Today, we explore the different properties of the region $\Omega = \{x : Ax \leq b\}$ – i.e. the region that our linear program maximizes over.

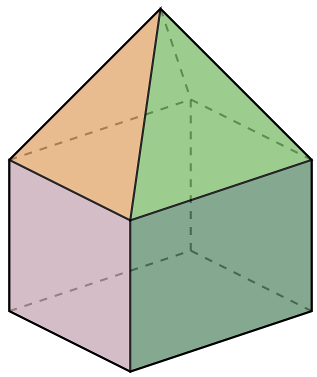


Figure 1: An example of a convex polytope. We can consider each face of the polytope as an affine inequality and then the polytope is all the points that satisfy each inequality. Notice that an affine inequality defines a half-plane and therefore is also the intersection of the half-planes.

(a) The first property that we will be interested in is *convexity*. We say that a space X is convex if for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in X.$$

That is, the entire line segment $\overline{xy}$ is contained in X. Prove that Ω is indeed convex.

(b) The second property that we will be interested in is showing that linear objective functions over convex polytopes achieve their maxima at the vertices. A vertex is any point $v \in \Omega$ such that v **cannot** be expressed as a point on the line $\overline{yz}$ for $v \neq y, v \neq z$, and $y, z \in \Omega$.

Prove the following statement: Let Ω be a convex space and f a linear function $f(x) = c^T x$. Show that the for a line $\overline{yz}$ for $y, z \in \Omega$ that $f(x)$ is maximized on the line at either y or z. I.e. show that

$$\max_{\lambda \in [0,1]} f(\lambda y + (1 - \lambda)z)$$

achieves the maximum at either $\lambda = 0$ or $\lambda = 1$. *Hint: Assume without loss of generality that* $f(y) \geq f(z)$.

(c) Now, prove that global maxima will be achieved at vertices. For simplicity, you can assume there is a unique global maximum.

Hint: Use the definition of a vertex presented above. (Side note: This argument is the basis of the Simplex algorithm by Dantzig to solve linear programs.)