Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Planting Trees

This problem will guide you through the process of writing a dynamic programming algorithm.

You have a garden and want to plant some apple trees in your garden, so that they produce as many apples as possible. There are n adjacent spots numbered 1 to n in your garden where you can place a tree. Based on the quality of the soil in each spot, you know that if you plant a tree in the ith spot, it will produce exactly x_i apples. However, each tree needs space to grow, so if you place a tree in the ith spot, you can't place a tree in spots $i-1$ or $i+1$. What is the maximum number of apples you can produce in your garden?

(a) Give an example of an input for which:

- Starting from either the first or second spot and then picking every other spot (e.g. either planting the trees in spots $1, 3, 5 \ldots$ or in spots $2, 4, 6 \ldots$) does not produce an optimal solution.
- The following algorithm does not produce an optimal solution: While it is possible to plant another tree, plant a tree in the spot where we are allowed to plant a tree with the largest x_i value.

(b) To solve this problem, we'll think about solving the following, more general problem: "What is the maximum number of apples that can be produced using only spots 1 to i?". Let $f(i)$ denote the answer to this question for any i. Define $f(0) = 0$, as when we have no spots, we can't plant any trees. What is $f(1)$? What is $f(2)$?

(c) Suppose you know that the best way to plant trees using only spots 1 to i does not place a tree in spot i. In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$. (Hint: What spots are we left with? What is the best way to plant trees in these spots?)

(d) Suppose you know that the best way to plant trees using only spots 1 to i places a tree in spot i. In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$.

(e) Describe a linear-time algorithm to compute the maximum number of apples you can produce. (Hint: Compute $f(i)$ for every i. You should be able to combine your results from the previous two parts to perform each computation in $O(1)$ time).

2 Change making

You are given an unlimited supply of coins of denominations $v_1, \ldots, v_n \in N$ and a value $W \in N$. Your goal is to make change for W using the minimum number of coins, that is, find a smallest set of coins whose total value is W.

(a) Design a dynamic programming algorithm for solving the change making problem. What is its running time?

(b) You now have the additional constraint that there is only one coin per denomination. Does your previous algorithm still work? If not, design a new one.

3 String Shuffling

Let x, y, and z be strings. We want to know if z can be obtained only from x and y by interleaving the characters from x and y such that the characters in x appear in order and the characters in y appear in order. For example, if $x =$ **prasad** and $y =$ **JOHN**, then it is true for $z =$ **praJOsHadN**, but false for $z =$ **prasadJOHNNY** (extra characters), $z =$ **praJOHsad** (missing the final **N**), and $z =$ **dasarpJOHN** (out of order). How can we answer this query efficiently? Your answer must be able to efficiently deal with strings with lots of overlap, such as $x =$ **aaaaaaaaaab** and $y =$ **aaaaaaaac**.

(a) Design an efficient algorithm to solve the above problem and state its runtime.

(b) Consider an iterative implementation of our DP algorithm in part (a). Naively if we want to keep track of every solved sub-problem, this requires $O(|x||y|)$ space (double check to see if you understand why this is the case). How can we reduce the amount of space our algorithm uses?

4 Egg Drop

You are given k identical eggs and an n story building. You need to figure out the highest floor $\ell \in \{0, 1, 2, \ldots n\}$ that you can drop an egg from without breaking it. Each egg will never break when dropped from floor ℓ or lower, and always breaks if dropped from floor $\ell + 1$ or higher. ($\ell = 0$ means the egg always breaks). Once an egg breaks, you cannot use it any more. However, if an egg does not break, you can reuse it.

Let $f(n, k)$ be the minimum number of egg drops that are needed to find ℓ (regardless of the value of ℓ).

(a) Find $f(1, k)$, $f(0, k)$, $f(n, 1)$, and $f(n, 0)$.

(b) Consider dropping an egg at floor x when there are n floors and k eggs left. Then, it either breaks, or doesn't break. In either scenario, determine the minimum remaining number of egg drops that are needed to find ℓ in terms of $f(\cdot, \cdot)$, n, k, and/or x.

(c) Find a recurrence relation for $f(n, k)$.

Hint: whenever you drop an egg, call whichever of the egg breaking/not breaking leads to more drops the "worst-case event". Since we need to find ℓ regardless of its value, you should assume the worst-case event always happens.