CS 170 Homework 13 (Optional)

1 Convexity Potpourri

For each of the following statements, state whether it is true or false. Justify your answers.

(a) The complement of a convex set $\mathcal{S} \in \mathbb{R}^d$ is always a convex set.

(b) The complement of a convex set $\mathcal{S} \in \mathbb{R}^d$ is never a convex set.

(c) The sum of two convex functions is a convex function.

(d) The difference of two convex functions is a convex function.

(e) The set of points (x, y) such that $x^2 + y^2 \geq 10$ is a convex set.

Solution:

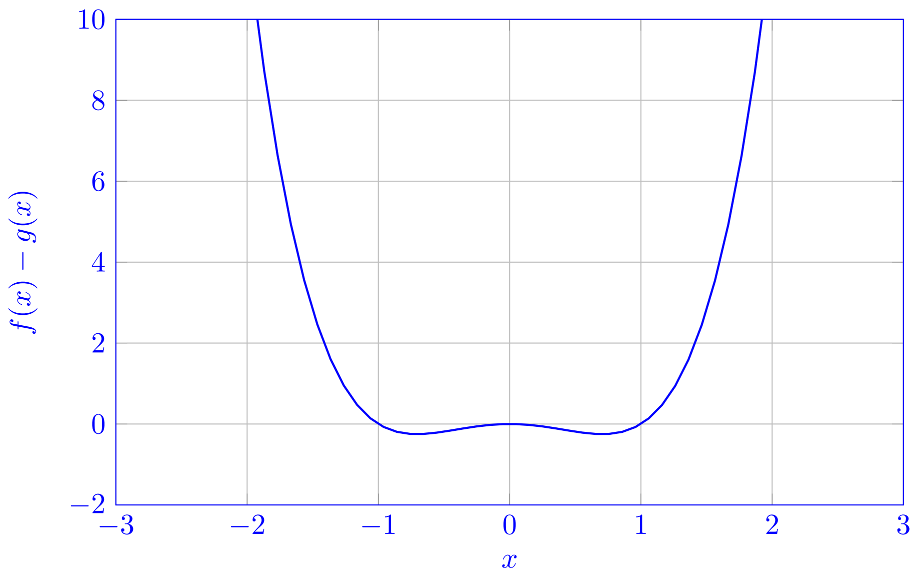
(a) False, consider $\mathcal{S}$ being a disk in 2D space. Although $\mathcal{S}$ is convex, its complement is not.

(b) False, consider letting $\mathcal{S}$ be a halfspace of $\mathbb{R}^d$. Halfspaces are convex, and the compliment of a halfspace is a halfspace.

(c) True. Let $f : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be convex functions. We have that

$$f\left(\frac{x+y}{2}\right) + g\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + g\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \frac{g(x)+g(y)}{2}$$

(d) False. Consider $f(x) = x^4$ and $g(x) = x^2$. $f(x) - g(x) = x^4 - x^2$ is not convex.



(e) False. Note that the equation $x^2+y^2 = 10$ describes a circle or disk. Hence, $x^2+y^2 \geq 10$ is the complement of the disk, which is not convex.

2 Fair Allocation

Consider the following problem:

FAIR ALLOCATION

There are N dollars to be allocated among n employees. Based on work experience, employee i is entitled at least ℓ_i dollars.

An allocation is defined to be fair if no subset of $n/10$ employees or fewer receives more than half the dollars.

Find a fair allocation if possible; otherwise, report that none exists.

(a) Write a linear program for the above problem (with possibly exponentially many constraints).

Solution: We can define a linear program with the following constraints. Note that since we are solving a feasibility problem, no objective function is needed.

Let x_i be the amount of money that employee i is allocated.

$$\begin{aligned} x_i &\geq \ell_i & \forall i \\ \sum_{x_i \in S} x_i &\leq \frac{N}{2} & \forall \text{ subsets } S \text{ of most } n/10 \text{ employees} \end{aligned}$$

The first set of constraints guarantee that each employee gets at least l_i dollars, and the second constraint guarantees that the allocation is fair.

(b) Describe a polynomial-time algorithm implementing the separation oracle[1] for the linear program from part (a).

Solution: Given some solution x, we can verify the first set of constraints in $O(n)$ time by just checking each individual constraint.

To determine if any of the fairness constraints are broken, simply check whether the $n/10$ highest paid employees are allocated to more than $\frac{N}{2}$ dollars. If they are, then we have found one such broken constraint. Otherwise, no fairness constraint is broken. This can be done in $O(n \log n)$ by sorting the employees in allocations and taking the $n/10$ highest paid. It can also be done in $O(n)$ by find the $n/10$-th highest paid employee using quickselect then iterating the employees to find the $n/10$ highest paid.

[1] The separation oracle a linear program determines a constraint that is violated by a point x, if there exists one.

3 Setting Tolls

Consider the following problem:

SETTING TOLLS

The city council has decided to impose tolls on some its most popular roads to discourage people from driving, aiming to reduce traffic and emissions. In particular, the goal is to discourage all driving from location s to location t in the city. The formal description of the problem is as follows.

Input: An undirected unweighted graph $G = (V, E)$, a pair of nodes s and t, and a positive real number B.

Output: A set of edge weights $w : E \to \mathbb{R}^+$ such that:

1. For every $s - t$ path P in the graph G, the total weight on the path is at least 1
2. The total weight of all edges in the graph is at most B

(weights w_e are the "tolls" on the edges)

(a) Write a linear program for the above problem (with possibly exponentially many constraints).

Solution: We can define a linear program with the following constraints. Note that since we are solving a feasibility problem, no objective function is needed.

$$\sum_{e \in p} w_e \geq 1 \qquad \forall p \text{ that is an } s - t \text{ path}$$
$$\sum_{e \in E} w_e \leq B$$

The first set of constraints guarantee that the total weight of all paths are at least 1. The second constraint guarantees that the total weight of all edges is at most B.

(b) Describe a polynomial-time algorithm implementing the separation oracle for the linear program.

Solution: The first set of constraints can be verified in $O(E \log V)$ time. Given a set of edge weights, w, run Dijkstra's algorithm from s to t. If the total weight of this path is less than 1, then we found a violated constraint. Otherwise, none of the constraints in the first set are violated.

The second constraint can be verified in $O(E)$ time by just summing the edge weights.