CS 170 Homework 10

Due **4/12/2023, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

2 Flow vs LP

You play a middleman in a market of m suppliers and n purchasers. The i-th supplier can supply up to $s[i]$ products, and the j-th purchaser would like to buy up to $b[j]$ products.

However, due to legislation, supplier i can only sell to a purchaser j if they are situated at most 1000 miles apart. Assume that you're given a list L of all the pairs (i, j) such that supplier i is within 1000 miles of purchaser j. Given m, n, $s[1..m]$, $b[1..n]$, and L as input, your job is to compute the maximum number of products that can be sold. The runtime of your algorithm must be polynomial in m and n.

For parts (a) and (b), assume the product is divisible—that is, it's OK to sell a fraction of a product.

(a) Show how to solve this problem, using a network flow algorithm as a subroutine. Describe the graph and explain why the output from the network flow algorithm gives a valid solution to this problem.

(b) Formulate this as a linear program. Explain why this correctly solves the problem, and the LP can be solved in polynomial time.

(c) Now let's assume you *cannot* sell a fraction of a product. In other words, the number of products sold by each supplier to each purchaser must be an integer. Which formulation would be better, network flow or linear programming? Explain your answer.

3 Reduction to 3-Coloring

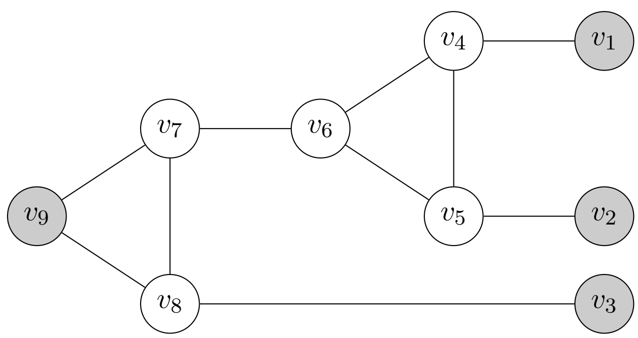
Given a graph $G = (V, E)$, a valid 3-coloring assigns each vertex in the graph a color from {red, green, blue} such that for any edge (u, v), u and v have different colors. In the 3-coloring problem, our goal is to find a valid 3-coloring if one exists. In this problem, we will give a reduction from 3-SAT to the 3-coloring problem. Since we know that 3-SAT is NP-Hard (there is a reduction to 3-SAT from every NP problem), this will show that 3-coloring is NP-Hard (there is a reduction to 3-coloring from every NP problem).

In our reduction, the graph will start with three special vertices, labelled v_{TRUE}, v_{FALSE}, and v_{BASE}, as well as the edges $(v_{\mathsf{TRUE}}, v_{\mathsf{FALSE}})$, $(v_{\mathsf{TRUE}}, v_{\mathsf{BASE}})$, and $(v_{\mathsf{FALSE}}, v_{\mathsf{BASE}})$.

(a) For each variable x_i in a 3-SAT formula, we will create a pair of vertices labeled x_i and $\neg x_i$. How should we add edges to the graph such that in any valid 3-coloring, one of $x_i, \neg x_i$ is assigned the same color as v_{TRUE} and the other is assigned the same color as v_{FALSE}?

Hint: any vertex adjacent to v_{BASE} must have the same color as either v_{TRUE} or v_{FALSE}. Why is this?

(b) Consider the following graph, which we will call a "gadget":



Consider any valid 3-coloring of this graph that does *not* assign the color red to any of the gray vertices (v_1, v_2, v_3, v_9). Show that if v_9 is assigned the color blue, then at least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Hint: it's easier to prove the contrapositive!

(c) We have now observed the following about the graph we are creating in the reduction:

(i) For any vertex, if we have the edges (u, v_{FALSE}) and (u, v_{BASE}) in the graph, then in any valid 3-coloring u will be assigned the same color as v_{TRUE}.

(ii) Through brute force one can also show that in a gadget, if all the following hold:

(1) All gray vertices are assigned the color green or blue.

(2) v_9 is assigned the color blue.

(3) At least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Then there is a valid coloring for the white vertices in the gadget.

Using these observations and your answers to the previous parts, **give a reduction from 3-SAT to 3-coloring. Prove that your reduction is correct (you do not need to prove any of the observations above).**

Hint: create a new gadget per clause!