Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

Flow. The *capacity* indicates how much flow can be allowed on an edge. Given a directed graph $G = (V, E)$ with edge capacites $c(u, v)$ (for all $(u, v) \in E$) and vertices $s, t \in V$, a flow is a mapping $f : E \to \mathbb{R}^+$ that satisfies

- Capacity constraint: $f(u, v) \leq c(u, v)$, the flow on an edge cannot exceed its capacity.
- Conservation of flows: $f^{\text{in}}(v) = f^{\text{out}}(v)$, flow in equals flow out for any $v \notin \{s, t\}$

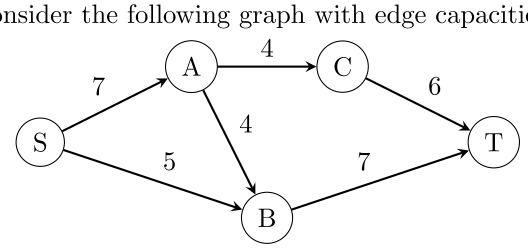
Here, we define $f^{\text{in}}(v) = \sum_{u:(u,v)\in E} f(u, v)$ and $f^{\text{out}}(v) = \sum_{u:(v,u)\in E} f(u, v)$. We also define $f(v, u) = -f(u, v)$, and this is called *skew-symmetry*. Note that the total flow in the graph is $\sum_{v:(s,v)\in E} f(s, v) = \sum_{u:(u,t)\in E} f(u, t)$, where s is the source node of the graph and t is the target node.

Residual Graph. Given a flow network (G, s, t, c) and a flow f, the *residual capacity* (w.r.t. flow f) is denoted by $c_f(u, v) = c_{uv} - f_{uv}$. And the *residual network* $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$.

Ford-Fulkerson. Keep pushing along s-t paths in the residual graph and update the residual graph accordingly. The runtime of this algorithm is $O(mF)$, where $m = |E|$ and F is the value of the max flow.

1 Residual in Graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

2 Secret Santa

Imagine you are throwing a party and you want to play Secret Santa. Thus you would like to assign to every person at the party another partier to whom they must anonymously give a single gift. However, there are some restrictions on who can give gifts to who: nobody should be assigned to give a gift to themselves or to their spouse. Since you are the host, you know all of these restrictions. Give an efficient algorithm that determines if you and your guests can play Secret Santa.

Canonical Form. The canonical form of a linear program is

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

where $x \geq 0$ means that every entry of the vector x is greater than or equal to 0.

Dual. The dual of the canonical LP is

$$\begin{aligned} \text{maximize } & b^\top y \\ \text{subject to } & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Weak duality: The objective value of any feasible dual $\leq$ objective value of any feasible primal.

Strong duality: The *optimal* objective values of these two are equal.
Both are solvable in polynomial time by the Ellipsoid or Interior Point Method.

3 Taking a Dual

Consider the following linear program:

$$\max 4x_1 + 7x_2$$
$$\text{s.t.} \begin{cases} x_1 + 2x_2 \leq 10 \\ 3x_1 + x_2 \leq 14 \\ 2x_1 + 3x_2 \leq 11 \\ x_1, x_2 \geq 0 \end{cases}$$

Construct the dual of the above linear program.

Zero Sum Games: In this game, there are two players: a maximizer and a minimizer. We generally write the payoff matrix M in perspective of the maximizer, so every row corresponds to an action that the maximizer can take, every column corresponds to an action that the minimizer can take, and a positive entry corresponds to the maximizer winning. M is a n by m matrix, where n is the number of choices the maximizer has, and m is the number of choices the minimizer has.

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,m} \end{bmatrix}$$

A linear program that represents fixing the maximizer's choices to a probabilistic distribution where the maximizer has n choices, and the probability that the maximizer chooses choice i is p_i is the following:

$$\begin{gathered} \max\ z \\ M_{1,1} \cdot p_1 + \cdots + M_{n,1} \cdot p_n \geq z \\ M_{1,2} \cdot p_1 + \cdots + M_{n,2} \cdot p_n \geq z \\ \vdots \\ M_{1,m} \cdot p_1 + \cdots + M_{n,m} \cdot p_n \geq z \\ p_1 + p_2 + \cdots + p_n = 1 \\ p_1, p_2, \ldots, p_n \geq 0 \end{gathered}$$

The dual represents fixing the minimizer's choices to a probabilistic distribution. If we let the probability that the minimizer chooses choice j be q_j, then the dual is the following:

$$\begin{gathered} \min\ w \\ M_{1,1} \cdot q_1 + \cdots + M_{1,m} \cdot q_m \leq w \\ M_{2,1} \cdot q_1 + \cdots + M_{2,m} \cdot q_m \leq w \\ \vdots \\ M_{n,1} \cdot q_1 + \cdots + M_{n,m} \cdot q_m \leq w \\ q_1 + q_2 + \cdots + q_m = 1 \\ q_1, q_2, \ldots, q_m \geq 0 \end{gathered}$$

By strong duality, the optimal value of the game is the same if you fix the minimizer's distribution first or the maximizer's distribution first.

4 Zero-Sum Games Short Answer

(a) Suppose a zero-sum game has the following property: The payoff matrix M satisfies $M = -M^\top$. What is the expected payoff of the row player?

(b) True or False: If every entry in the payoff matrix is either 1 or -1 and the maximum number of 1s in any row is k, then for any row with less than k 1s, the row player's optimal strategy chooses this row with probability 0. Justify your answer.

(c) True or False: Let M_i denote the ith row of the payoff matrix. If $M_1 = \frac{M_2+M_3}{2}$, then there is an optimal strategy for the row player that chooses row 1 with probability 0.

5 Domination

In this problem, we explore a concept called *dominated strategies*. Consider a zero-sum game with the following payoff matrix for the row player:

		Column: A	B	C
Row:	D	1	2	-3
	E	3	2	-2
	F	-1	-2	2

(a) If the row player plays optimally, can you find the probability that they pick D without directly solving for the optimal strategy? Justify your answer.

(Hint: How do the payoffs for the row player picking D compare to their payoffs for picking E?)

(b) Given the answer to part a, if the both players play optimally, what is the probability that the column player picks A?? Justify your answer.

(c) Given the answers to part a and b, what are both players' optimal strategies?

Note: All parts of this problem can be solved without using an LP solver or solving a system of linear equations.