CS 170 Homework 11

Due **Monday 4/15/2024, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

2 Some Sums

Given an array $A = [a_1, a_2, \ldots, a_n]$ of nonnegative integers, consider the following problems:

1 **Partition**: Determine whether there is a subset $S \subseteq [n]$ ($[n] := \{1, 2, \cdots, n\}$) such that $\sum_{i \in S} a_i = \sum_{j \in ([n] \setminus S)} a_j$. In other words, determine whether there is a way to partition A into two disjoint subsets such that the sum of the elements in each subset equal.

2 **Subset Sum**: Given some integer x, determine whether there is a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = x$. In other words, determine whether there is a subset of A such that the sum of its elements is x.

3 **Knapsack**: Given some set of items each with weight w_i and value v_i, and fixed numbers W and V, determine whether there is some subset $S \subseteq [n]$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq V$.

For each of the following clearly describe your reduction and justify its correctness.

(a) Find a linear time reduction from Subset Sum to Partition.

(b) Find a linear time reduction from Subset Sum to Knapsack.

3 k-XOR

In the k-XOR problem, we are given n boolean variables $x_1, x_2, \ldots, x_n$, a list of m clauses each of which is the XOR of exactly k distinct variables (that is, the clause is true if and only if an odd number of the k variables in the clause are true), and an integer c. Our goal is to decide if there is some assignment of variables that satisfies at least c clauses.

(a) In the Max-Cut problem, we are given an undirected unweighted graph $G = (V, E)$ and integer α and want to find a cut $S \subseteq V$ such that at least α edges cross this cut (i.e. have exactly one endpoint in S). Give and argue correctness of a reduction from Max-Cut to 2-XOR.

Hint: every clause in 2-XOR is equivalent to an edge in Max-Cut.

(b) Give and argue correctness of a reduction from 3-XOR to 4-XOR.

4 Survivable Network Design

Survivable Network Design is the following problem:

> We are given two $n \times n$ matrices: a cost matrix d_{ij} and a connectivity requirement matrix r_{ij}, both of which are symmetric. We are also given a budget b. Our goal is to find an undirected graph $G = (\{1, ..., n\}, E)$ such that the total cost of all edges (i.e. $\sum_{(i,j) \in E} d_{ij}$) is at most b and there are exactly r_{ij} edge-disjoint paths between any two distinct vertices i and j; if no such G exists, output "None". (A set of paths is edge-disjoint if no edge appears in more than one of them)

Show that Survivable Network Design is NP-Complete.

Hint: Reduce from a NP-Hard problem in Section 8 of the textbook.

5 Orthogonal Vectors

In the 3-SAT problem, we have n variables and m clauses, where each clause is the OR of (at most) three of these variables or their negations. The goal of the problem is to find an assignment of variables that satisfies all the clauses, or correctly declare that none exists.

In the orthogonal vectors problem, we have two sets of vectors A, B. All vectors are in $\{0,1\}^m$, and $|A| = |B| = n$. The goal of the problem is to find two vectors $a \in A, b \in B$ whose dot product is 0, or correctly declare that none exists. The brute-force solution to this problem takes $O(n^2 m)$ time: compute all $|A||B| = n^2$ dot products between two vectors in A, B, and each dot product takes $O(m)$ time.

Show that if there is a $O(n^c m)$-time algorithm for the orthogonal vectors problem for some $c \in [1, 2)$, then there is a $O(2^{cn/2} m)$-time algorithm for the 3-SAT problem. For simplicity, you may assume in 3-SAT that the number of variables must be even.

Hint: Try splitting the variables in the 3-SAT problem into two groups.