## Problem solving.

By yourself: $O(n^2)$

Two of you: divide and conquer.

<table>
<thead>
<tr>
<th>What's the point of all this?</th>
</tr>
</thead>
<tbody>
<tr>
<td>You are already problem solvers!</td>
</tr>
<tr>
<td>We are giving tools to be more precise ... and powerful (with recursion and programming and algorithms.)</td>
</tr>
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More divide and conquer: mergesort.

<table>
<thead>
<tr>
<th>Mergesort(A)</th>
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<tr>
<td>if (length(A) &gt;1)</td>
</tr>
<tr>
<td>return (merge(mergesort(a[1],...,a[n/2]), mergesort(a[n/2+1],...,a[n])))</td>
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<tr>
<td>else</td>
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<td>return a</td>
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<th>Sorted SubArray 1:</th>
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<tr>
<td>3, 7, 8, 10, 11...</td>
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<th>Sorted Subarray 2:</th>
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<tr>
<td>4, 5, 9, 19, 20,...</td>
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3, 4, 5, 7, 8 
...

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<td>Split the objects into a larger and smaller group.</td>
</tr>
<tr>
<td>One sorts the larger one, the other the small one.</td>
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<tr>
<td>Put together.</td>
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<td>Divide and conquer.</td>
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<th>How do you sort a set of objects?</th>
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<td>In high school. Before you came to Berkeley.</td>
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<td>How would you sort by yourself?</td>
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<td>Talk to your neighbor!</td>
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<td>Do this $n$ times.</td>
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<td>Split the objects in half.</td>
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<td>Each of you sort.</td>
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<td>Merge the sorted piles.</td>
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Mergesort: running time analysis

Mergesort(A)
if (length(A) > 1)
    return
    merge(mergesort( a[1],...,a[n/2]),
          mergesort( a[n/2+1],...,a[n]))
else
    return a

Split: $O(n)$ time
Could be $O(1)$, e.g., MergeSort(A,start,finish).
Merge: each element in output takes one comparison : $O(n)$.
Recursive: 2 subproblems of size $n/2$.

$T(n) = 2T(n/2) + O(n)$

Masters: $T(n) = aT(n/b) + O(n^d)$
with $\log_b a = d \implies O(n^{\log_b a})$

Apply Masters:
$a = 2, b = 2, d = 1 \implies \log_2 2 = 1 \implies T(n) = O(n\log n)$

Sorting lower bound.

Thm: Comparison sort requires $\Omega(n\log n)$ comparisons.

Proof idea: Input: $a_1,a_2,...,a_n$
Possible Output: $a_b,a_c,...,a_{15}$
Represent output as permutation of $[1,...,n]$.
Output: $8,n-8,...,15$.

How many possible outputs? $n!$
Algorithm must be about to output any of $n!$ permutations.
Algorithm must output just 1 permutation at termination.

Check it out...

Iterative Mergesort: Bottom up, use of queues.
Make each element into list and put lists in queue.

| 1 | 0 | ... |

Merge first two lists, put in queue (at end).

| 1 | 0 | ... |

Rinse. Repeat.

| 1 | 0 | ... |

And next pass through queue...

| 1 | 0 | ... |

Each pass through queue: each element touched once. $O(n)$ time.
Each pass halves number of lists.

$\implies O(\log n)$ passes $\implies O(n\log n)$ time

Sorting lower bound: ...proof

Algorithm must be able to output any of $n!$ permutations.
Algorithm must output just 1 permutation at termination.

Algorithm as tree of comparisons.
After a sequence of comparisons get to termination or 1 permutation.

$S$ is set of possible permutations at some point in Algorithm
Example: After no comparisons, any output is possible.

Do some comparison: $a_i > a_j$?
If Yes, Alg "could" return subset of permutations: $S_1$.
If No, Alg "could" return subset of permutations: $S_2$.

$S_1 \cup S_2 = S \implies \max(|S_1|, |S_2|) \geq |S|/2$.

Each comparison divides possible outputs by at most 2.
Need at least $\log_2 (n!)$ comparisons to get to just 1 permutation.
...to get to termination.

$m \geq \log_2 (n!) \implies \log n! = \Omega(\log(n^2)) = \Omega(n\log n)$.

Figure for proof.

Either the set of permutations $S_1$ or $S_2$ is larger.
One must be at least half.
Depth must be $\Omega(\log(\# permutations)) = \Omega(\log(n!)) = \Omega(n\log n)$.

Can we do better than mergesort? Yes? No?
No. For comparison sort.
(Recall from 61b: radix sort may be faster: $O(n)$.)

A research area: "bit complexity" versus "word complexity".
Radix sort and lower bound.

Why is radix sort not subject to lower bound?
It buckets by ‘letter’.
So, the degree of the tree is not 2. It is larger.

Solve a harder Problem: Selection.

For a set of n items S.
Select kth smallest element.
Median: select ⌊ n/2 ⌋ + 1 elt.

Select(k, S).

Find the median element of a set of elements: a₁, …, aₙ.
Median is value, v, where ⌈ n/2 ⌉ elts are less than v (if n is odd.)
Versus Average?
Average household income (2004): $70,700
Median household income (2004): $43,200
Why so different? Bill Gates and Jeff Bezos. The 1%, perhaps.
Why use average?
Find average? Compute \( \frac{\text{sum}}{n} \).
Median finding.

Solve a harder Problem: Selection.

For a set of n items S.
Select kth smallest element.
Median: select ⌊ n/2 ⌋ + 1 elt.
Example.
k = 7 for items {11, 48, 5, 21, 2, 15, 17, 19, 15}
Output?
(A) 19
(B) 15
(C) 21

(???)

Correctness:
On average?
Will eventually return 19, which is 7th element of list.
Correct in all cases.

The Induction.

Base Case: k = 1, |S| = 1. Trivial.

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<tr>
<th>S_L</th>
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<th>S_R</th>
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<tr>
<td>11</td>
<td>48</td>
<td>5</td>
</tr>
<tr>
<td>21</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>19</td>
<td>15</td>
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If \( k \leq |S_L| \), Select(k, S_L)
\( k \)th element in first \( |S_L| \) elts.
kth elt of \( S \) is \( k \)th elt of \( S_L \).
else \( k > |S_L| + |S_L| + |S_L| = |S_L| \).
\( k \)th elt of \( S \) is \( |S_L| + |S_L| = |S_L| \).
\( k \)th elt of \( S \) is \( |S_L| = |S_L| \).
else Select(k – |S_L| – |S_L|, S_R)
kth elt is in \( S_R \) and
\( k \)th elt of \( S \) is \( k – |S_L| – |S_L| \) after els of \( S_L \) & \( S_L \).
Correct in all cases.

Selection: runtime.

Worst case runtime?
(A) \( \Theta(n \log n) \)
(B) \( \Theta(n) \)
(C) \( \Theta(n^2) \)
Let \( k = n \).
Partition element is smallest every time.
Size of list decrease by 1 in each recursive call.
Time for partition is \( O(i) \) time when / elements.
\( \Theta(n + (n – 1) + \cdots + 2 + 1) = \Theta(n^2) \) time. or (C)
Worse than sorting!
On average?
Average time to get a heads?

Flip a coin, what is average number of tosses to get a heads?
(A) two
(B) three
(C) Could go forever!
(A) ...and (C) (but not relevant.)

Randomized sorting method: quicksort.

Choose a random element to be partition elt, p.
Split into larger and smaller elements based on p.
Recurse on each group.
\( T(n) = ??? \)

How do you analyse this?
Master Theorem doesn’t apply...
Also, how does one do average time?
Expected Time of algorithm on \( n \) elements.

Conclusion: More divide and conquer.

You came up with iterative, and divide and conquer algorithms.
Iterative: \( O(n^2) \).
Mergesort: \( O(n \log n) \).
Divide in two, sort each, merge.
Median: \( O(n) \) expected time.
Do selection.
Pick random partition, recurse on the correct side.
Analysis: after an average of two steps, problem smaller by 3/4.
Geometric series of runtimes.
Quicksort: \( O(n \log n) \) expected time.
Random partition.
Work per/element is constant in each iteration.
Expected \( O(\log n) \) iterations.
\( O(n \log n) \) work for \( n \) elements.

Expected (average) Time?

Lemma: Expected number of coin tosses to get a heads is 2.

Proof:
\[
E[X] = 1 + \frac{1}{2}E[X] \\
\Rightarrow E[X] = 2. \quad \square
\]

Probability that random pivot elt in the middle half is \( \geq \frac{1}{2} \).

Expected time to get a middle element is \( E[X] \times O(n) = O(n) \).
Pick in the middle half subproblem size is \( \leq \frac{3}{4}n \).

Expected time recurrence:
\[
T(n) \leq T\left(\frac{3}{4}n\right) + O(n). \]

Masters or just thinking: \( n + (3/4)n + (3/4)^2n + \cdots = O(n) \)
\[ \Rightarrow T(n) = O(n). \]

Other sorting method.

Find median.
Partition elements into halves according to median.
Sort each half.
\( T(n) = 2T(n/2) + O(n) \).
\[ T(n) = O(n \log n). \]

One method: amortized analysis.

From the point of view of an element.
Begin: size of elt’s group is \( n \).
Each iteration: size of elt’s group gets smaller.
Size of the group become \( \leq 3n/4 \) if ...
an elt in middle fourth is partition elt.
\( X \) - number of iterations for group to be small enough.
We know \( X \sim O(\frac{1}{2}) \).
We did this before.
\[ E[X] = 1/2 \times (1 + E(X)) + 1/2 \times (1) \]
\[ \text{or } E[X] = 2. \]

How much expected work on this element?
\[ E[n] = 2 + E[3/4n] \]
Unfold: \[ E[n] = 2 + 2 + \cdots + 2 \]
The number of 2’s is \( \log_{1/4} n \).
So, \( O(\log n) \) per element, or \( O(n \log n) \) overall elements.