Problem solving.

How do you sort a set of objects?

Problem solving.

How do you sort a set of objects?

In high school.

Problem solving.

How do you sort a set of objects?

In high school. Before you came to Berkeley.

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In high school. Before you came to Berkeley.

How would you sort by yourself?

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How would you sort by yourself?

Talk to your neighbor!

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How would you sort with two of you?

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Do some work splitting up?

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How would you sort with two of you?

  Do some work splitting up?

  Do some work putting together?

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How do you sort a set of objects?

In high school. Before you came to Berkeley.

How would you sort by yourself?

Talk to your neighbor!

How would you sort with two of you?

Do some work splitting up?

Do some work putting together?

Talk to your neighbor.

By yourself: $O(n^2)$

Pick the largest one,
  set it aside,
  sort the rest,
  and put the largest last.

By yourself: $O(n^2)$

Pick the largest one,
set it aside,
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Choose the largest one: $O(n)$.

By yourself: $O(n^2)$

Pick the largest one,
set it aside,
sort the rest,
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Choose the largest one: $O(n)$.

Do this n times.

By yourself: $O(n^2)$

Pick the largest one,
set it aside,
sort the rest,
and put the largest last.

Choose the largest one: $O(n)$.

Do this n times.

$O(n^2)$.

Two of you: divide and conquer.

Split the objects in half.

Two of you: divide and conquer.

Split the objects in half.

Each of you sort.

Two of you: divide and conquer.

Split the objects in half.

Each of you sort.

Merge the sorted piles.

Two of you: divide and conquer.

Split the objects in half.

Each of you sort.

Merge the sorted piles.

Divide and conquer.

Two of you: divide and conquer.

Split the objects into a larger and smaller group.

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Split the objects into a larger and smaller group.

One sorts the larger one, the other the small one.

Two of you: divide and conquer.

Split the objects into a larger and smaller group.

One sorts the larger one, the other the small one.

Put together.

Two of you: divide and conquer.

Split the objects into a larger and smaller group.

One sorts the larger one, the other the small one.

Put together.

Divide and conquer.

What's the point of all this?

You are already problem solvers!

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We are giving tools to be more precise ...

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You are already problem solvers!

We are giving tools to be more precise ...
and powerful (with recursion and programming and algorithms.)

More divide and conquer: mergesort.

Sort items in n elt array: $A = [a_1, \ldots, a_n]$,

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Sort items in n elt array: $A = [a_1, \ldots, a_n]$,
E.g., $A = [5, 6, 7, 9, 10, 2, 3 ...]$.

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Mergesort(A)

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if (length(A) >1)
   return
      (merge(mergesort(a[1],...,a[n/2]),
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How to merge?

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Choose lowest from two lists,

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How to merge?

Choose lowest from two lists, cross out,

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How to merge?

Choose lowest from two lists, cross out, repeat.

Sorted SubArray 1: $3\ ,7\ ,8\ ,10, 11, \ldots$
Sorted Subarray 2: $4\ ,5\ ,9, 19, 20, \ldots$

, , , ,

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Mergesort: running time analysis

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Split: $O(n)$ time

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Could be $O(1)$, e.g., **MergeSort(A,start,finish)**.

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Merge: each element in output takes one comparision

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$$T(n) = 2T(\frac{n}{2}) + O(n).$$

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Masters: $T(n) = aT(n/b) + O(n^d)$
with $\log_b a = d$

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Apply Masters:
$a = 2$, $b = 2$, $d = 1$

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Apply Masters:
$a = 2$, $b = 2$, $d = 1 \implies \log_2 2 = 1$

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Apply Masters:

$a = 2$, $b = 2$, $d = 1 \implies \log_2 2 = 1 \quad \implies T(n) = O(n \log n)$.

Check it out...

Iterative Mergesort: Bottom up, use of queues.

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Make each element into list and put lists in queue.

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$\boxed{8}$ $\boxed{3}$ $\boxed{5}$ $\boxed{9}$ $\cdots\cdots\cdot$

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Make each element into list and put lists in queue.

$$\boxed{\boxed{8}\quad\boxed{3}\quad\boxed{5}\quad\boxed{9}\qquad\cdots\cdots\cdot\qquad}$$

Merge first two lists, put in queue (at end).

$$\boxed{\boxed{5}\quad\boxed{9}\qquad\cdots\cdots\cdot\qquad\boxed{3,8}\quad}$$

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$\boxed{5}\ \boxed{9}\ \cdots\cdots\ \boxed{3,8}$

Rinse.

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Rinse. Repeat.

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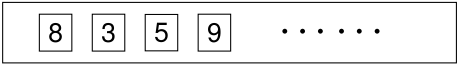

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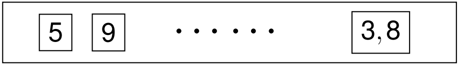
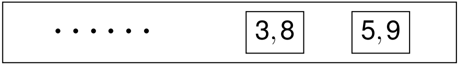


Rinse. Repeat.



And next pass through queue...

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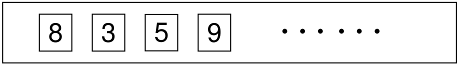

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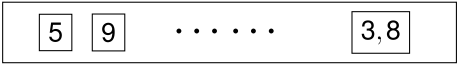

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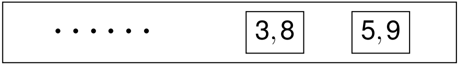

And next pass through queue...

Each pass through queue:

Check it out...

Iterative Mergesort: Bottom up, use of queues.

Make each element into list and put lists in queue.

| 8 | 3 | 5 | 9 | ······· |

Merge first two lists, put in queue (at end).

| 5 | 9 | ······· | 3, 8 |

Rinse. Repeat.

| ······· | 3, 8 | 5, 9 |

And next pass through queue...

| ······· | 3, 5, 8, 9 |

Each pass through queue: each element touched once.

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Each pass through queue: each element touched once. $O(n)$ time.

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Each pass through queue: each element touched once. $O(n)$ time.
Each pass halves number of lists.

Check it out...

Iterative Mergesort: Bottom up, use of queues.

Make each element into list and put lists in queue.

$\boxed{8}\ \boxed{3}\ \boxed{5}\ \boxed{9}\ \cdots\cdots\cdot$

Merge first two lists, put in queue (at end).

$\boxed{5}\ \boxed{9}\ \cdots\cdots\cdot\ \boxed{3,8}$

Rinse. Repeat.

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$\Longrightarrow$ $O(\log n)$ passes $\Longrightarrow$ $O(n \log n)$ time

Sorting lower bound.

Can we do better?

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Comparison sorting algorithm only compares numbers.

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Does not look at bits only uses result of comparison.

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Compare two first elts and then output first.

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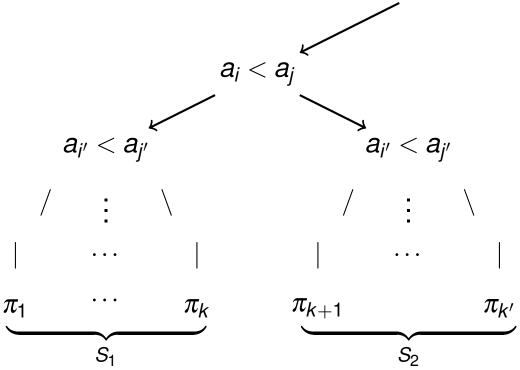
Figure for proof.



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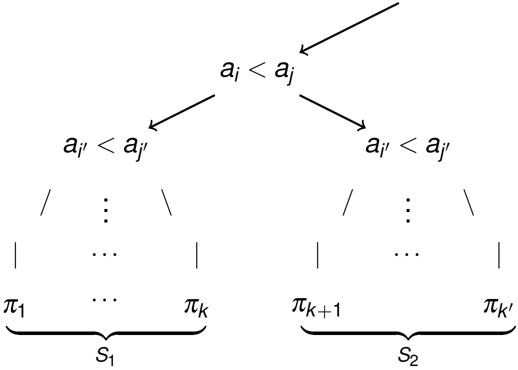

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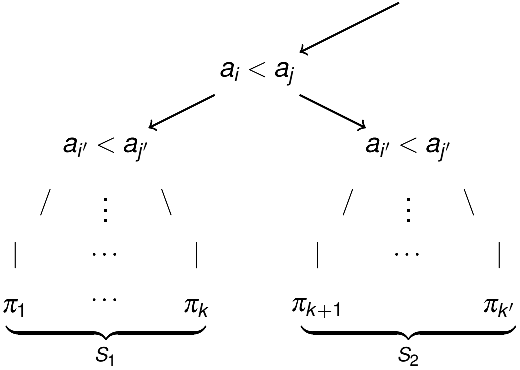

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One must be at least half.

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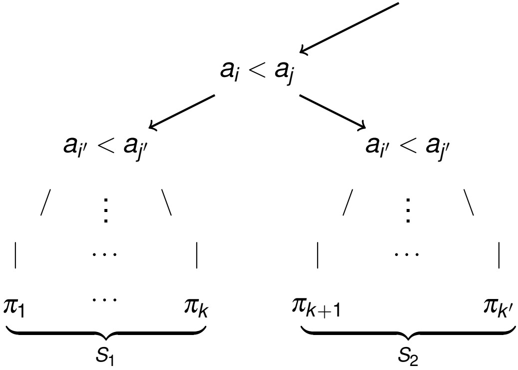

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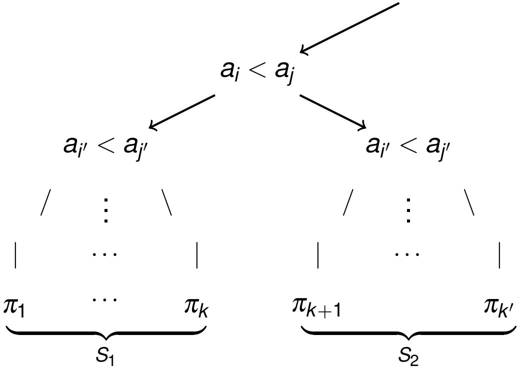

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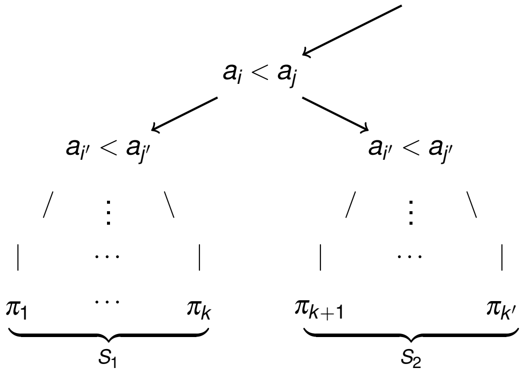

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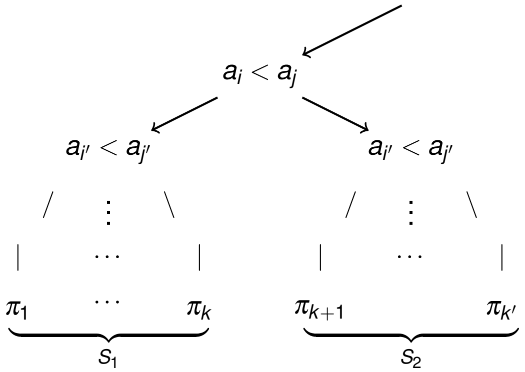

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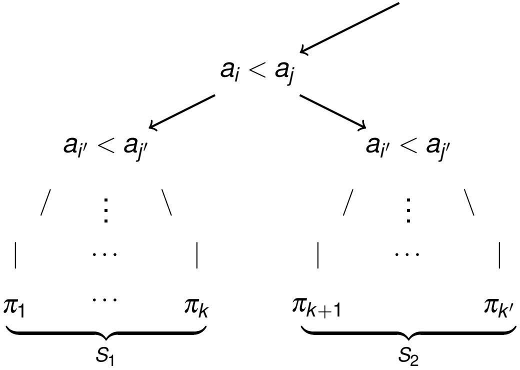

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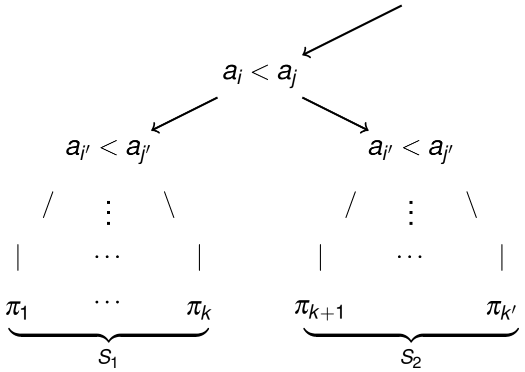

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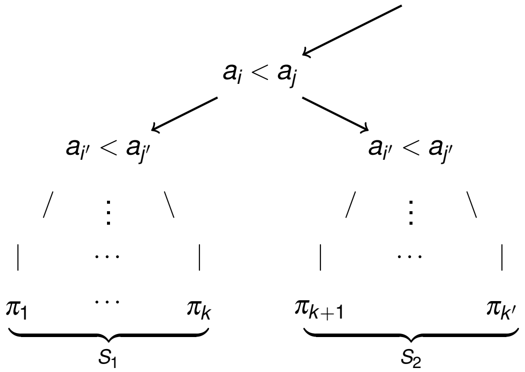

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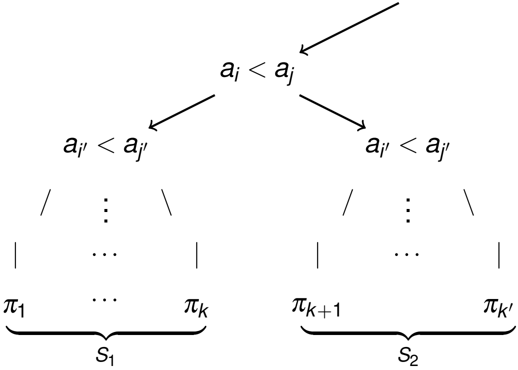

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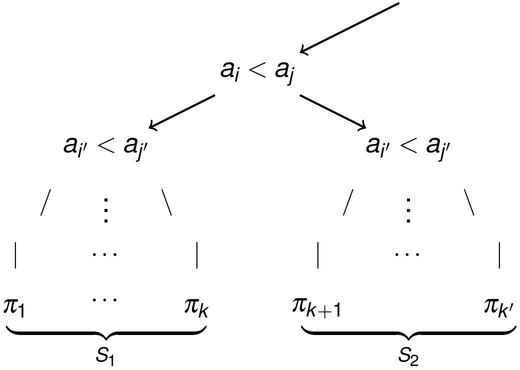

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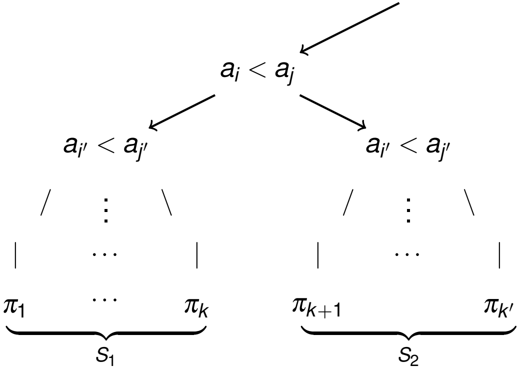

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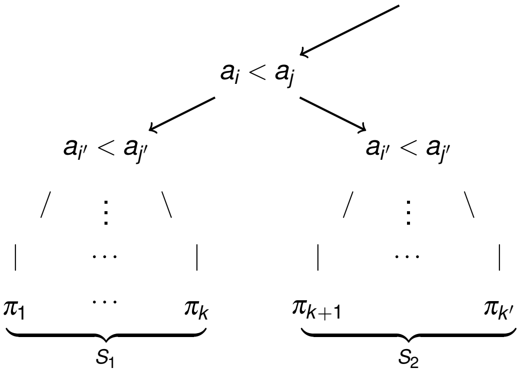

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A research area: “bit complexity” versus “word complexity”.

Radix sort and lower bound.

Why is radix sort not subject to lower bound?

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So, the degree of the tree is not 2. It is larger.

Median finding.

Find the median element of a set of elements: $a_1, \ldots, a_n$.

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Average household income (2004): $70,700

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Better algorithm?

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Form S_v containing all elts $= v$		$S_v: 15, 15$
Form S_R containing all elts $> v$		$S_R: 48, 21, 17, 19$
If $k \leq \lvert S_L \rvert$, Select(k, S_L).		$7 \leq 3$?
elseif $k \leq \lvert S_L \rvert + \lvert S_v \rvert$, return v.		$7 \leq 5$?
else Select$(k - \lvert S_L \rvert - \lvert S_v \rvert, S_R)$		Select$(2, [48, 21, 17, 19])$

Will eventually return 19, which is 7th element of list.

Correctness: Induction.
Idea: Subroutine returns correct answer, and so will I !
Base case is good. Subroutine calls ..by design.

The Induction.

Base Case: $k = 1$, $|S| = 1$. Trivial.

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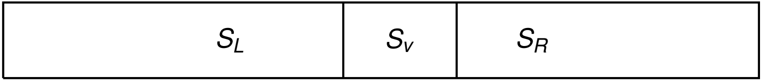

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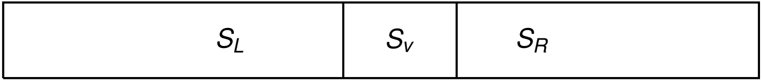

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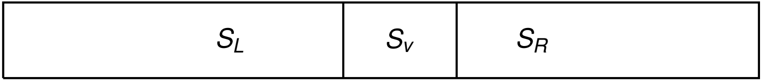

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On average?

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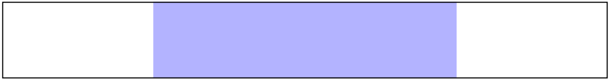

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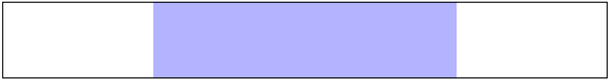

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Other sorting method.

Find median.

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Partition elements into halves according to median.

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Randomized sorting method: quicksort.

Choose a random element to be partition elt, p.
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Expected Time of algorithm on n elements.

One method: amortized analysis.

From the point of view of an element.

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Begin: size of elt's group is n.

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Unfold: $E[n] = 2 + 2 + \cdots + 2$

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So, $O(\log n)$ per element,

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an elt in middle fourth is partition elt.

X - number of iterations for group to be small enough.
We know $X \sim G(\frac{1}{2})$.
We did this before.

$E[X] = 1/2 \times (1 + E(X)) + 1/2 \times (1)$
or $E[X] = 2$.

How much expected work on this element?
$E[n] = 2 + E[3/4n]$
Unfold: $E[n] = 2 + 2 + \cdots + 2$
The number of 2's is $\log_{4/3} n$.
So, $O(\log n)$ per element, or $O(n \log n)$ overall elements.

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$O(n \log n)$ work for n elements.